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Open localizations

F. Borceux and M. Korostenski

Département de Mathématiques, Université Catholique de Louvain, 2, Chemin du Cyclotron, 1348 Louvain-La-Neuve, Belgium

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Abstract

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An essential localization of a finitely complete category $\mathscr{C}$ is a full reflective subcategory $\mathscr{D}$ whose reflection has itself a left adjoint. $\mathscr{D}$ can then be seen as a $\mathscr{C}$-indexed category and the localization is open when the adjunctions are indexed ones. We give equivalent conditions for the openness of a localization and prove that, with good conditions on $\mathscr{C}$, open localizations constitute a locale.

1. Introduction

Since some time, following an idea that F.W. Lawvere calls the "unity of opposites", interest has been put in the study of essential localizations. Given a finitely complete category $\mathscr{C}$, a localization $\mathscr{D}$ of it is a full reflective subcategory, saturated for isomorphisms and whose reflection is left exact. The localization is essential when the reflection has itself a left adjoint, which turns out to be automatically full and faithful. We refer to [11] for what concerns essential localizations.

When $\mathscr{C}$ is a topos, special attention is paid to the 'open localizations' of $\mathscr{C}$; they are special instances of essential localizations. A localization of a topos is just a subtopos of it, thus is characterized by a topology $j : \Omega \to \Omega$ on the subobject classifier. The open subtoposes are those for which the topology j has a left adjoint. In the case of a topos of sheaves on a space X, they are in one-to-one correspondence with the open subsets of X (cf. [8]).

It has been proved in [5] that a localization of a locally presentable category $\mathscr{C}$ is completely determined by a Lawvere–Tierney topology on a subobject classifier, in some bigger topos canonically associated with the category. The open localizations of $\mathscr{C}$ are those for which the corresponding topology has a left adjoint. That condition is equivalent to the existence of a universal dense interior operation for the universal closure operation induced by the localization on the lattices of subobjects. It is also equivalent to the essentialness of the localization together with an exactness condition relating the two left adjoints.

Even for a presheaf topos $\mathscr{C}$, it was noticed in [11] that the intersection of two essential localizations is generally not an essential localization. But the intersection of two open localizations of any locally presentable category $\mathscr{C}$ is again an open localization. It is also proved in [11] that the supremum of a family of essential localizations of a complete and cocomplete category exists and is just the usual supremum in the poset of localizations. When the individual localizations are open, this supremum is again open provided $\mathscr{C}$ is a locally presentable category where unions are universal; the same assumption on $\mathscr{C}$ implies that its open localizations constitute a locale. This result is somewhat amazing due to the fact that when finite limits commute with filtered colimits, localizations constitute *the dual* of a locale (cf. [3]). Let us also mention that when moreover arbitrary unions are effective in $\mathscr{C}$ (cf. [1]), an explicit formula can be given for describing the supremum of a family of open localizations. And when colimits are universal in $\mathscr{C}$, the locale of open localizations reduces to the locale of subobjects of the terminal object $\mathbf{1}$.

When $\mathscr{C}$ is a topos, the reader will notice that our definition of open localization reduces also to that of a "molecular subtopos" (cf. [2]). Following the presentation of [2], we could thus also say that an open localization $\mathscr{D}$ of $\mathscr{C}$ is a $\mathscr{C}$-indexed essential localization of $\mathscr{C}$.

2. Open localizations

Let $\mathscr{C}$ be a locally α-presentable category, where α is some regular cardinal. We denote by $\mathbb{P}$ the full subcategory of α-presentable objects. $\mathscr{C}$ is equivalent to α-Cont($\mathbb{P}^{\mathrm{op}}$, $\mathscr{S}ets$), the category of α-continuous presheaves on $\mathbb{P}$. We denote by $\hat{\mathbb{P}}$ the topos of presheaves on $\mathbb{P}$. The topos $\hat{\mathbb{P}}$ has a subobject classifier Ω given by

$$\Omega(P) = \text{set of subpresheaves of } \mathbb{P}(-, P)\,,$$

where the action on the arrows of $\mathbb{P}$ is given by pulling back. The subobject classifier for $\mathscr{C}$ is the subobject of Ω given by

$$\Omega_{\mathscr{C}}(P) = \text{set of } \alpha\text{-exact subpresheaves of } \mathbb{P}(-, P)\,;$$

$\Omega_{\mathscr{C}}$ is a $\wedge$-semi lattice in the topos $\hat{\mathbb{P}}$.

Given a localization $l \dashv i : \mathscr{D} \leftrightarrows \mathscr{C}$ of $\mathscr{C}$, the corresponding universal closure operation on the lattices of subobjects in $\mathscr{C}$ is given by the following pullback

$$\begin{tikzcd}
\bar{S} \arrow[r] \arrow[d, tail] & ilS \arrow[d, tail] \\
A \arrow[r, "\eta_A"'] & ilA
\end{tikzcd}$$

where η_A is the unit of the adjunction and $S \rightarrowtail A$ is any subobject in $\mathscr{C}$. The monomorphisms which are dense for this closure operation are exactly those which are inverted by the reflection l. Moreover, the corresponding topology $j_{\mathscr{D}} : \Omega_{\mathscr{C}} \to \Omega_{\mathscr{C}}$ applies, at each level, a subobject on its closure (cf. [5]).

The following result has been noticed by Kelly and Lawvere (cf. [11]):

Proposition 1. *If $k \dashv l \dashv i : \mathscr{D} \leftrightarrows \mathscr{C}$ is an essential localization of $\mathscr{C}$, every object A in $\mathscr{C}$ has a dense interior.*

Proof. Consider the subobject

$$\mathring{A} = \bigcap \{S \mid S \text{ is a dense subobject of } A\}\ .$$

Since l preserves arbitrary intersections, $\mathring{A}$ is still a dense subobject of A and thus is its dense interior. $\square$

The main result of this section is the following theorem:

Theorem 2. *Let $l \dashv i : \mathscr{D} \leftrightarrows \mathscr{C}$ be a localization of the locally presentable category $\mathscr{C}$. The following conditions are equivalent:*

(1) *The corresponding closure operation admits a universal dense interior operation.*

(2) *The associated topology $j_{\mathscr{D}} : \Omega_{\mathscr{C}} \to \Omega_{\mathscr{C}}$ has a left adjoint.*

(3) *The localization is essential and such that, if the first diagram below is a pullback in $\mathscr{D}$, so is the second diagram in $\mathscr{C}$.*

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{p.b.}"] & D' \arrow[d, "g"] \\
lC \arrow[r, "lc"'] & lC'
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
kD \arrow[r, "kd"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{p.b.}"] & kD' \arrow[d, "\bar{g}"] \\
C \arrow[r, "c"'] & C'
\end{tikzcd}$$

Proof. (1)$\Rightarrow$(2) The left adjoint $\sigma : \Omega_{\mathscr{C}} \to \Omega_{\mathscr{C}}$ to the topology $j_{\mathscr{D}}$ applies, at each level, a subobject on its dense interior. The universality of the dense interior operation implies the naturality of σ and the adjointness property is obvious.

(3)$\Rightarrow$(1) Since for every $C \in \mathscr{C}$, one has $lc \cong lklC$, the assumption applied to the following obvious situation

$$\begin{tikzcd} lC \arrow[r, equal] \arrow[d, "\eta_{lC}"'] \arrow[dr, phantom, "\text{p.b.}"] & lC \arrow[d, equal] \\ lklC \arrow[r, "l\varepsilon_C"'] & lC \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} klC \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\text{p.b.}"] & klC \arrow[d, "\varepsilon_C"] \\ klC \arrow[r, "\varepsilon_C"'] & C \end{tikzcd}$$

implies that ε_C is a monomorphism. ε_C is in fact a dense monomorphism, since it is inverted by l. Now if $S \rightarrowtail C$ is another dense subobject of C, the isomorphism $lC \cong lS$ yields by adjunction a morphism $klC \to S$, proving finally that klC is the dense interior of C.

(2)$\Rightarrow$(3) A subobject $S \rightarrowtail C$ in $\mathscr{C}$ has a characteristic mapping $\varphi_S : C \to \Omega_{\mathscr{C}}$ in $\hat{\mathbb{P}}$. If $\sigma : \Omega_{\mathscr{C}} \to \Omega_{\mathscr{C}}$ is the left adjoint to the topology $j_{\mathscr{D}}$, the composite $j_{\mathscr{D}} \circ \varphi_S$ is the characteristic mapping of some subobject $\mathring{S} \rightarrowtail C$ in $\mathscr{C}$. Using the adjunction $\sigma \dashv j_{\mathscr{D}}$, one deduces immediately that $\mathring{S}$ is the dense interior of S. This applies in particular to the subobject C itself and yields the required universal dense interior operation. Clearly, this operation induces an endofunctor on $\mathscr{C}$.

To conclude the proof of this last implication, we shall need some lemmas. For the sake of brevity, we shall not recall every time the assumptions, which are those at the present stage of the proof. C will always denote an object in $\mathscr{C}$.

Lemma 3. *The 'dense interior' endofunctor on $\mathscr{C}$ preserves binary products.*

Proof. If $i : \mathring{C} \rightarrowtail C$ is the canonical inclusion of $\mathring{C}$ in C, the following pullback

$$\begin{tikzcd} \mathring{C} \times C \arrow[r, "p_1"] \arrow[d, tail, "i \times \mathrm{id}"'] & \mathring{C} \arrow[d, tail, "i"] \\ C \times C \arrow[r, "p_1"'] & C \end{tikzcd}$$

proves that $(C \times C)^\circ \cong \mathring{C} \times C$; in the same way one proves that $(C \times C)^\circ \cong C \times \mathring{C}$. Then

$$(C \times C)^\circ \cong (\mathring{C} \times C) \cap (C \times \mathring{C}) \cong \mathring{C} \times \mathring{C}\,. \quad \square$$

Lemma 4. *The 'dense interior' endofunctor on $\mathscr{C}$ transforms every monomorphism into a closed monomorphism.*

Proof. If $S \rightarrowtail C$ is a subobject in $\mathscr{C}$, consider the following pullbacks, where the closure operation is that on the lattice of subobjects of C.

$$\begin{tikzcd}
\mathring{S} \arrow[r, tail, "\text{dense}"] \arrow[d, tail, "\text{dense}"'] & \mathring{\bar{S}} \arrow[r, tail, "\text{closed}"] \arrow[d, tail, "\text{dense}"] & \mathring{C} \arrow[d, tail, "\text{dense}"] \\
S \arrow[r, tail, "\text{dense}"'] & \bar{S} \arrow[r, tail, "\text{closed}"'] & C
\end{tikzcd}$$

Since density and closedness are preserved both by pulling back and composition, we conclude that $\mathring{S}$ is dense in $\bar{S}$, thus is bigger than $\mathring{\bar{S}}$; therefore, $\mathring{S} \cong \mathring{\bar{S}}$, which proves the lemma. □

Lemma 5. *The unit of the adjunction $\eta_C : C \to ilC$ is a monomorphism precisely when the diagonal of C is closed.*

Proof. This is a classical result (cf. [1]). □

Lemma 6. *The objects $C \in \mathscr{C}$ and $lC \in \mathscr{D}$ have the same dense interior.* □

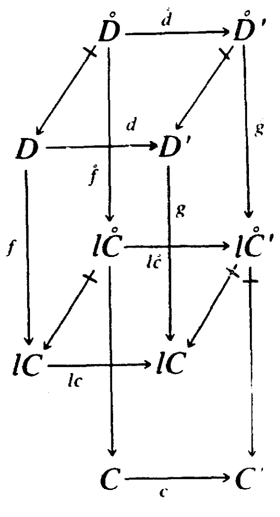
Let us see the functor i as being just a canonical embedding. Consider the following pullback, where η_C is the unit of the adjunction $l \dashv i$:

$$\begin{tikzcd}
\mathring{C} \arrow[r] \arrow[d, tail] & il\mathring{C} \arrow[d] \\
C \arrow[r, "\eta_C"'] & ilC
\end{tikzcd}$$

By Lemma 4, the interior of the diagonal $\mathring{\Delta} : \mathring{C} \rightarrowtail (C \times C)^\circ$ is closed, thus the diagonal $\Delta : \mathring{C} \rightarrowtail \mathring{C} \times \mathring{C}$ is closed as well (Lemma 3) and the canonical morphism $\eta_{\mathring{C}} : \mathring{C} \to il\mathring{C}$ is a monomorphism. But $l\mathring{C} \cong lC$, which implies that the composite morphism $\mathring{C} \to ilC$ in the previous pullback is a monomorphism. So the morphism $\mathring{C} \to il\mathring{C}$ is a monomorphism as well; it is in fact a dense monomorphism as pullback of the morphism η_C which is inverted by the left exact reflection. So $\mathring{C}$ is a dense subobject of ilC and thus contains $il\mathring{C}$; this implies the required isomorphism. □

We are now ready to conclude the proof of the theorem. First of all, the 'dense interior' functor, restricted to the subcategory $\mathscr{D}$, produces a functor $k = (\mathring{\ }) \circ i : \mathscr{D} \to \mathscr{C}$ which is left adjoint to l. Indeed every subobject in the subcategory $\mathscr{D}$ is closed, thus equal to its dense interior; this allows us to choose the identity as being the unit of the adjunction. In view of Lemma 6, the canonical inclusion $\mathring{C} \to C$ can be chosen as the counit $il\mathring{C} \to C$ of the adjunction.

It remains to verify the compatibility condition on pullbacks. It derives from the consideration of the following diagram:



All faces of the cube are pullbacks, except the back face; so this back face is a pullback too. The lower vertical square is a pullback whose existence is asserted by Lemma 6; combining it with the back face of the cube yields the required condition. □

Definition 7. An '*open localization*' of a locally presentable category is one which satisfies the equivalent conditions of Theorem 2.

Condition (2) of Theorem 2 shows that the open localizations of a topos are precisely its open subtoposes (cf. [10]). In fact, an essential localization $k \dashv l \dashv i : \mathscr{D} \rightleftarrows \mathscr{C}$ is in particular a colocalization $k \dashv l : \mathscr{D} \rightleftarrows \mathscr{C}$, since k is full and faithful (cf. [6]). This colocalization is completely determined by the endofunctor $k \circ l : \mathscr{C} \to \mathscr{C}$, since $\mathscr{D}$ is equivalent to the category of fixed points of this endofunctor. In the case of an open localization, condition (3) in Theorem 2 can be applied to the following pullbacks, where $\mathbf{1}$ is the terminal object of $\mathscr{C}$.

$$\begin{tikzcd} lC \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\text{p.b.}"] & \mathbf{1} \arrow[d, equal] \\ lC \arrow[r] & l\mathbf{1} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} klC \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & k\mathbf{1} \arrow[d] \\ C \arrow[r] & \mathbf{1} \end{tikzcd}$$

This yields the relation $klC \cong C \times k\mathbf{1}$. Since we have proved that $k\mathbf{1} \cong kl\mathbf{1} \cong \mathring{\mathbf{1}}$, an open localization is thus completely determined by a subobject $k\mathbf{1} \rightarrowtail \mathbf{1}$; the comonad associated with the corresponding colocalization is just the product by $k\mathbf{1}$. Such a comonad can always be defined from every subobject of $\mathbf{1}$, but has no reason in general to be induced by an open localization.

3. Finite infima of open localizations

The observant reader will notice that the results of this section are valid for a category $\mathscr{C}$ with pullbacks, using (3) in Theorem 2 as the definition of an open localization.

First of all let us notice that the identity on $\mathscr{C}$ is always an open localization, from which the existence of empty intersections. It remains to prove the existence of binary intersections. We refer to [3] for what concerns intersections of arbitrary localizations.

When $k \dashv l \dashv i : \mathscr{D} \rightleftarrows \mathscr{C}$ is an open localization of a topos $\mathscr{C}$ of sheaves on a space X, $\mathscr{D}$ is just the topos of sheaves on some open subset U. k extends a sheaf F from U to X by defining $F(V) = \emptyset$ when $V \not\subseteq U$; l is just the obvious restriction functor and i extends a sheaf F from U to X by putting $F(V) = F(V \cap U)$. The infimum of the two localizations corresponding to open subsets U and V is just the localization corresponding to $U \cap V$. Having in mind this example, it will be more intuitive to treat the problem considering k (and not i) as a canonical inclusion.

Proposition 8. *Let $\mathscr{C}$ be a locally presentable category. The infimum of two open localizations of $\mathscr{C}$, in the lattice of localizations of $\mathscr{C}$, is still an open localization.*

Proof. Consider two open localizations $k \dashv l \dashv i : \mathscr{D} \rightleftarrows \mathscr{C}$ and $k' \dashv l' \dashv i' : \mathscr{D}' \rightleftarrows \mathscr{C}$ of a locally presentable category $\mathscr{C}$. We view k and k' as canonical inclusions; if necessary, we saturate $\mathscr{D}$ and $\mathscr{D}'$ for the isomorphisms, along these inclusions, and we define k'' to be the inclusion of $\mathscr{D} \cap \mathscr{D}'$ in $\mathscr{C}$. The dense interior operations corresponding to the original localizations are just $k \circ l$ and $k' \circ l'$. Considering the following pullback diagram for an object $C \in \mathscr{C}$

$$\begin{tikzcd}
klC \cap k'l'C \arrow[r, rightarrowtail] \arrow[d, rightarrowtail] & klC \arrow[d, rightarrowtail] \\
k'l'C \arrow[r, rightarrowtail] & C
\end{tikzcd}$$

we conclude that $kl(k'l'C) = klC \cap k'l'C = k'l'(klC)$. Therefore, $klC \cap k'l'C$ is in $\mathscr{D} \cap \mathscr{D}'$ and it makes sense to define $l''(C) = klC \cap k'l'C$; this definition extends obviously to a functor $l'' : \mathscr{C} \to \mathscr{D}$.

Consider $\mathscr{D} \in \mathscr{D} \cap \mathscr{D}'$, $C \in \mathscr{C}$ and a morphism $f'' : k''D \to C$. Since $k''D = kD$, f'' corresponds to a morphism $f : D \to lC$; in the same way we get a morphism $f' : D \to l'C$ and thus finally a factorization $\bar{f} : D \to l''C$ (recall k, k' and k'' are canonical inclusions). From this it follows easily that l'' is right adjoint to k''.

We must also construct a right adjoint i'' to l''. Since $l'' = k \circ l \circ k' \circ l'$, it follows immediately that i'' will be given by the formula $i'' = i' \circ l' \circ i \circ l$. This is the same as

$i'' = i' \circ l' \circ i$ since $\mathcal{D} \cap \mathcal{D}'$ is stable for l. Equivalently one could have used the formula $i'' = i \circ l \circ i'$.

Now consider the two localizations $l \dashv i : \mathcal{D} \leftrightarrows \mathcal{C}$ and $l' \dashv i' : \mathcal{D} \leftrightarrows \mathcal{C}$. To prove that the localization $l'' \dashv i'' : \mathcal{D} \cap \mathcal{D}' \leftrightarrows \mathcal{C}$ is the infimum of the two original localizations in the lattice of all localizations, it suffices obviously to show that $i\mathcal{D} \cap i\mathcal{D}' = i''(\mathcal{D} \cap \mathcal{D}')$. The formula $i'' = i' \circ l' \circ i$ shows that $i''(\mathcal{D} \cap \mathcal{D}') \subseteq \mathcal{D}'$ and in the same way $i''(\mathcal{D} \cap \mathcal{D}') \subseteq \mathcal{D}$; thus $i''(\mathcal{D} \cap \mathcal{D}') \subseteq i\mathcal{D} \cap i\mathcal{D}'$. Conversely, if $iD \cong C \cong i'D'$, we have

$$i''D \cong i'l'D \cong i'l'C \cong i'l'i'D' \cong i'D' \cong C$$

so that $i\mathcal{D} \cap i\mathcal{D}' \subseteq i''(\mathcal{D} \cap \mathcal{D}')$.

Finally, $k''l''$ is obviously the universal dense interior operation associated with this infimum, since so are kl and $k'l'$ for the two individual open localizations. □

4. Suprema of open localizations

In a locally presentable category $\mathcal{C}$, the universality of unions is certainly a rather strong property. It is clearly satisfied when colimits are universal, thus in particular in every Grotendieck quasi-topos (cf. [4]), but those conditions are by no means necessary. For example, if $\mathbb{P}$ is a small category, consider for $\mathcal{C}$ the category of monomorphism preserving functors from $\mathbb{P}$ to the category $\mathcal{S}ets$ of sets. This category can be presented as that of those covariant presheaves which satisfy an axiom

$$f(a) = f(a') \Rightarrow a = a'$$

for each monomorphism $f : A \rightarrowtail B$ in $\mathbb{P}$ and each pair a, a' of variables of type A; it is therefore a locally presentable category (cf. [7]). In this category $\mathcal{C}$, unions are obviously computed as in the category of all covariant presheaves on $\mathbb{P}$, so they are universal. But coequalizers in $\mathcal{C}$ are generally not universal; this is already the case when $\mathbb{P}$ is the category $\mathbf{2} = \{\bullet \rightrightarrows \bullet\}$.

Proposition 9. *In a locally presentable category $\mathcal{C}$ where unions are universal, the supremum of a family of open localizations, computed in the lattice of localizations, is again an open localization.*

Proof. Let us consider a family $k_m \dashv l_m \dashv i_m : \mathcal{D}_m \leftrightarrows \mathcal{C}$ of essential localizations; as in Section 3, we view the functors k_m as canonical inclusions. Since $\mathcal{C}$ is complete and cocomplete, the supremum in the lattice of localizations is in fact an essential localization $k \dashv l \dashv i : \mathcal{D} \leftrightarrows \mathcal{C}$ and the morphisms inverted by l are exactly those inverted by each individual l_m (cf. [11]).

Given $C \in \mathscr{C}$, consider the union $\bigcup k_m l_m C \rightarrowtail C$. We mentioned already that a subobject $C' \rightarrowtail C$ is l-dense precisely when it is j_m-dense for each individual index m; this is equivalent to C' containing each subobject $k_m l_m C$, thus to C' containing the union $\bigcup k_m l_m C$. Therefore, $\bigcup k_m l_m C$ is the l-dense interior of C.

The universality of unions, joined to the universality of each individual $k_m l_m$-interior operation, implies clearly the universality of the kl-interior operations. □

Theorem 10. *In a locally presentable category $\mathscr{C}$ where unions are universal, the open localizations constitute a locale.*

Proof. Consider the situation of the previous proposition as well as an additional open localization $k' \dashv l' \dashv i' : \mathscr{D}' \rightleftarrows \mathscr{C}$. For an object $C \in \mathscr{C}$, we must prove the isomorphism

$$k'l'C \cap \left(\bigcup k_m l_m C\right) \cong \bigcup (k'l'C \cap k_m l_m C)$$

as subobjects of C (cf. Propositions 8 and 9). This is just another instance of the universality of unions. □

Let us mention that when unions are effective in $\mathscr{C}$, the supremum referred to in Proposition 9 can be explicitly described. We recall that the union $\bigcup S_m \rightarrowtail C$ of a family of subobjects is effective when each family of morphisms $f_m : S_m \to C'$ which pairwise agree on each intersection $S_m \cap S_n$, factors uniquely through the union $\bigcup S_m$. This is just an obvious extension of the notion of binary effective union introduced in [1]. Notice that in the case of a finitely presentable category $\mathscr{C}$, filtered unions are already universal; so the universality of all unions reduces to the strictness of the initial object and the universality of binary unions.

So let us consider the situation of Proposition 9, with the additional assumption that unions are effective. There is no restriction to suppose that the given family of localizations is hereditary, since this does not change the supremum; this is just to make sure that the infimum of any two given localizations is already in the family. $\mathscr{D}$ is thus the full subcategory of those objects $C \in \mathscr{C}$ such that $\bigcup k_k l_m C$ is the whole of C and k is the canonical inclusion; lC is just $\bigcup k_m l_m C$. When a localization $\mathscr{D}$ is smaller than a localization $\mathscr{D}'$, $\mathscr{D}$ is stable for $i' \circ l'$ so that we get a natural transformation $i' \circ l' \Rightarrow i \circ l \circ i' \circ l' \cong i \circ l$. For an object $D \in \mathscr{D}$, let us define $i(D)$ to be the limit in $\mathscr{C}$ of the diagram constituted of the various objects $i_m \circ l_m(D)$ and the connecting morphisms we have just defined between them. Given $C \in \mathscr{C}$, a morphism $f : C \to iD$ corresponds to a compatible family of morphisms $f_m : C \to i_m l_m D$, thus by adjunction to compatible families of morphisms $f'_m : l_m C \to l_m D$ and finally $f''_m : k_m l_m C \to D$. By effectiveness of the unions, this last family corresponds exactly to a morphism $f'' : \bigcup k_m l_m C \to D$, which shows that i is indeed right adjoint to l.

To conclude, let us observe the following:

Proposition 11. *When colimits are universal in the locally presentable category $\mathscr{C}$, the open localizations of $\mathscr{C}$ constitute a locale isomorphic to the locale of subobjects of the terminal object $\mathbf{1}$ of $\mathscr{C}$.*

Proof. If colimits are universal, so are unions and Theorem 10 applies. Now going back to the considerations concluding Section 2, we know that an open localization $k \dashv l \dashv i : \mathscr{D} \rightleftarrows \mathscr{C}$ is completely characterized by the subobject $kl\mathbf{1} \rightarrowtail \mathbf{1}$.

Conversely, if $U \rightarrowtail \mathbf{1}$ is an arbitrary subobject, define $\mathscr{D}$ to be the full subcategory of those objects C such that $C \times U \cong C$. Thus C is in $\mathscr{D}$ precisely when the unique morphism $C \to \mathbf{1}$ factors through U. In other words, $\mathscr{D}$ can be presented as the slice category $\mathscr{C}/U$. We have thus a coreflection $k \dashv l : \mathscr{D} \rightleftarrows \mathscr{C}$, where k is acting by composition with $U \rightarrowtail C$ and l by pullback along this same morphism. Since colimits are universal in $\mathscr{C}$, the pullback functor l preserves them and therefore has a right adjoint, since we are dealing with locally presentable categories (cf. [7]). So we have got an essential localization of $\mathscr{C}$ and it does obviously satisfy the exactness condition described in Theorem 2(3).

It is immediate that those two constructions are mutual inverses. □

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